J. Opt. A: Pure Appl. Opt. **5** (2003) L9–L11 PII: S1464-4258(03)57569-9

LETTER TO THE EDITOR

Direct evidence of negative refraction at media with negative ϵ and μ

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Received 12 December 2002, in final form 3 March 2003
Published 31 March 2003
Online at stacks.iop.org/JOptA/5/L9

Abstract
We analyse wave scattering by an interface between a positive refractive index medium and a medium for which both the permittivity and the permeability are negative. We show that the negative refraction effect can be deduced directly from Maxwell equations, without references to causality or losses.

Keywords: Light propagation, optical constants, beam optics

There has recently been a huge interest in the physics of electromagnetic wave propagation in media where the permeability and the permittivity in the harmonic regime both have a negative real part. Such materials have been considered by Veselago [1], but his work was a dead letter until it was recently suggested that they could be realized as effective media in the low frequency limit of heterogeneous materials [2–5]. These media are sometimes called 'left handed' though the terminology is confusing with respect to chiral media. Here we shall use the term 'Veselago medium' or 'Veselago material'. There is no *a priori* condition on the sign of the real parts of the permeability and the permittivity, in contrast to their imaginary parts, where energy conservation implies a positive sign, provided that the time dependence be chosen as $\exp(-i\omega t), \omega > 0$ [6]. Some spectacular applications have already been proposed [7], though the subject is a very polemical one [8–11]. It is not our point here to enter these polemical issues, but to try to clarify a certain number of very basic points in the scattering theory of such a medium. In particular, we address the crucial problem of the radiation conditions in such media. By a direct computation, involving only the Maxwell system in its distributional form, we compute the behaviour of a beam illuminating a homogeneous slab of Veselago materials, and show the asymptotic form of the transmitted beam.

Let us consider the basic scattering problem of a plane wave illuminating a plane interface separating the vacuum (ε_0, μ_0) from a Veselago material with relative parameters (ε, μ). The only difficulty here is the problem of the outgoing wave condition inside the Veselago medium. Indeed, let us assume s-polarized waves and a z-independent plane wave, with wavenumber k_0, illuminating the plane interface under the incidence θ. Then, denoting $\alpha = k_0 \sin\theta$, the total electric field is written $u(x)e^{i\alpha y}\boldsymbol{e}_z$ and satisfies

$$\frac{d}{dx}\left(\frac{1}{\mu}\frac{du}{dx}\right) + \left(k_0^2\varepsilon - \frac{\alpha^2}{\mu}\right)u = 0 \qquad (1)$$

in the Schwartz distribution meaning. Denoting

$$\beta = \sqrt{k_0^2\varepsilon\mu - \alpha^2} \qquad \beta_0 = \sqrt{k_0^2 - \alpha^2}, \qquad (2)$$

we impose, for $x < 0$, the radiation condition: $u = e^{i\beta_0 x} + r(\alpha, k_0)e^{-i\beta_0 x}$. But for $x > 0$, the situation should be discussed. In [13] it is claimed that at the interface between a regular and Veselago medium 'negative refraction follows immediately from the continuity of the tangential component of k and normal component of (Poynting vector)'; this last statement is false (it suffices to make the direct computation to see that there are both incident and reflected components in the Poynting vector), but in fact, the main point at issue is the choice of a radiation condition, something that cannot be deduced from Maxwell equations in the case of a plane interface, in contrast to the transmission conditions.

First, we only have two possible choices for the transmitted wave: $t(\alpha, k_0)e^{i\alpha x}e^{-i\beta x}$ or $t(\alpha, k_0)e^{i\alpha x}e^{i\beta x}$ (indeed, we necessarily have the conservation of the horizontal component of the wavevector). Usually, the second solution is kept, as representing a plane wave moving from the interface. However, in a Veselago material, the Poynting vector $P = \frac{1}{2\omega\mu}\boldsymbol{k}$ is directed in the opposite sense to $\boldsymbol{k}$. It seems that it is mainly on this basis that the choice is made to reverse the usual convention and choose $t(\alpha, k_0)\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-\mathrm{i}\beta x}$ as a transmitted field, and then obtain a non-usual transmission condition (negative index). It is not immediate that this is a good choice, because, although we would like energy to flow from the interface, we would also like to have retarded and not advanced waves.

One possible justification, which we recall briefly [12], is mathematical: we have to choose a branch of the square root to define function β. In the case of usual materials, and for a time dependence of $\exp(-\mathrm{i}\omega t)$, the chosen branch is determined by $\sqrt{1} = 1$ and a cut-line along $\mathrm{i}\mathbb{R}^-$ for instance. Here, we have to consider $(\varepsilon, \mu) \to \beta(\varepsilon, \mu)$ as a function of two complex variables. In order to define this function one possibility is to use the above defined square root and write function β as the composition of two functions:

$$(\varepsilon, \mu) \to k_0^2\varepsilon\mu - \alpha^2 \overset{\sqrt{}}{\to} \beta(\varepsilon, \mu); \tag{3}$$

this definition is not the only one for defining function β. However, it has the advantage of using one sole square root for every problem. Let us start from the point $(1, 1)$ where we know that $\beta(1, 1) = \cos\theta$. If the couple (ε, μ) then describes half a circle in the upper complex plane, then the phase of the product $\varepsilon\mu$ describes the entire interval $[0, 2\pi]$ so that $z = \varepsilon\mu$ describes a circle and thus crosses the cut-line, so that after this crossing, the image point $(\varepsilon\mu, \beta(\varepsilon, \mu))$ moves on the second sheet of the Riemann surface of $z \to z^2$ so of course when $(\varepsilon, \mu) = (-1, -1)$ we have by continuity that $\beta(-1, -1) = -\beta(1, 1)$. This is precisely the situation that one wants to avoid in the theory of multi-defined functions, and hence the introduction of a cut-line. However, it is not desirable, from a mathematical point of view, to write $\sqrt{1 \times 1} = 1$ and $\sqrt{(-1) \times (-1)} = -1$ because it means that we do not use a properly defined function. So a possible way out is to assume that there are some losses in the Veselago medium: we write $\mu(\omega) \simeq -1$, and $\varepsilon(\omega) = -1 + \mathrm{i}\eta(\omega)$. Then $\beta \simeq \pm(1 - \frac{\mathrm{i}\eta}{2})$, and to prevent the exponential growth of waves we have to choose the $-$ determination. This can be realized in a perfectly coherent way by defining the square root with a cut-line on $\mathbb{R}^+$, $\sqrt{\mathrm{i}} = (1 + \mathrm{i})\frac{\sqrt{2}}{2}$, the function being defined by upper continuity on $\mathbb{R}^+$ (i.e. $\sqrt{x} = \sqrt{x + \mathrm{i}0^+}$).

Direct numerical simulations have given evidence for negative refraction [12, 14]. Here, we propose a justification given directly by Maxwell equations, which does not require the use of losses or considerations on branches of the square root. We start with a situation where there is no possible ambiguity: we consider a homogeneous slab of width h illuminated by a plane wave. The slab is filled with an Veselago medium (ε, μ) and it is surrounded by the vacuum. This situation is very simple, for there is no problem of outgoing wave conditions in that case other than in vacuum. The Maxwell system in the distribution meaning allows us to obtain, in the same conditions as (1), (2),

$$\begin{aligned} x < 0 &: u(x) = \mathrm{e}^{\mathrm{i}\beta_0 x} + r\mathrm{e}^{-\mathrm{i}\beta_0 x} \\ 0 < x < h &: u(x) = A\mathrm{e}^{\mathrm{i}\beta x} + B\mathrm{e}^{-\mathrm{i}\beta x} \\ x > h &: u(x) = t\mathrm{e}^{\mathrm{i}\beta_0(x-h)} \end{aligned} \tag{4}$$

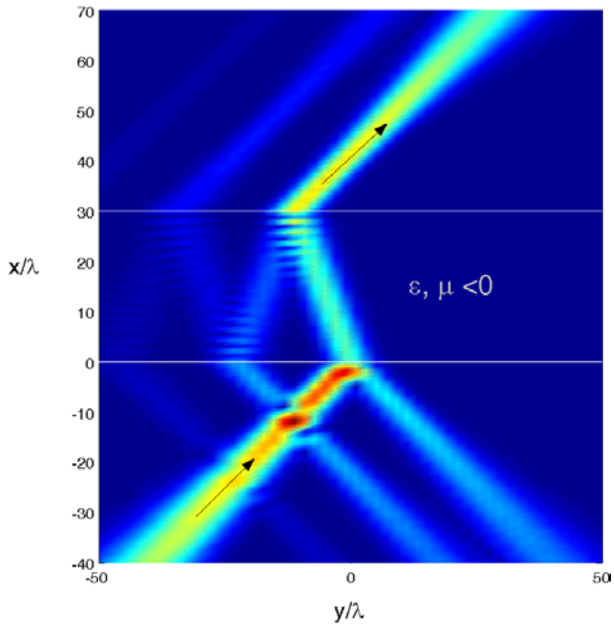

Figure 1. Map of the electric field for an incident monochromatic Gaussian beam (the half-space of incidence is $x < 0$).
(This figure is in colour only in the electronic version)

where

$$A = \frac{1}{2}\left(1 + \mu\frac{\beta_0}{\beta}\right)t\mathrm{e}^{-\mathrm{i}\beta h} \qquad B = \frac{1}{2}\left(1 - \mu\frac{\beta_0}{\beta}\right)t\mathrm{e}^{\mathrm{i}\beta h}$$
$$t = \frac{(\kappa^2 - 1)\mathrm{e}^{\mathrm{i}\beta h}}{\kappa^2 - \mathrm{e}^{2\mathrm{i}\beta h}} \qquad r = \frac{\kappa(\mathrm{e}^{2\mathrm{i}\beta h} - 1)}{\kappa^2 - \mathrm{e}^{2\mathrm{i}\beta h}} \tag{5}$$

and

$$\kappa = \frac{\beta + \mu\beta_0}{\beta - \mu\beta_0}. \tag{6}$$

We note that β is real because both ϵ and μ are negative. Let us assume now that it is a monochromatic wavepacket that illuminates the slab, with some spectral amplitude $\mathcal{A}(\alpha)$. Then we get, as a transmitted field,

$$u_t = \int \mathcal{A}(\alpha)t(k_0, \alpha)\mathrm{e}^{\mathrm{i}(\alpha y + \beta_0 x)}\,\mathrm{d}\alpha. \tag{7}$$

The important point now is to note that for ϵ and $\mu < 0$, $|\kappa| < 1$ in contrast to what happens for μ and $\epsilon > 0$. Therefore we get the celebrated series representing multiscattering inside the slab [2]:

$$t = (\kappa^2 - 1)\frac{-\mathrm{e}^{-\mathrm{i}\beta h}}{1 - \kappa^2\mathrm{e}^{-2\mathrm{i}\beta h}} = (1 - \kappa^2)\mathrm{e}^{-\mathrm{i}\beta h}\sum_p \kappa^{2p}\mathrm{e}^{-2\mathrm{i}p\beta h} \tag{8}$$

with a minus sign in the exponential terms.

Note that this series and its physical interpretation are direct consequences of Maxwell equations.

We recover the usual result that the transmitted field is a collection of rays, whose first one is given by

$$u_{t,0}(0, y) = \int \mathcal{A}(\alpha)(1 - \kappa^2)\mathrm{e}^{\mathrm{i}(\alpha y - \beta h)}\,\mathrm{d}\alpha. \tag{9}$$

However, the minus sign in front of β implies that the barycentre of the beam is displaced towards the left and not towards the right as usual (with the grating convention that θ is positive in the direct orientation): there is a negative beam refraction.

Inside the slab, the same expansion can be obtained for A and B, and the first transmitted and reflected beams are

$$u^{+}(x, y; h) = \int \mathcal{A}(\alpha)\frac{1}{2}\left(1 + \mu\frac{\beta_0}{\beta}\right)(1 - \kappa^2)\mathrm{e}^{\mathrm{i}(\alpha y - 2\beta h)}\mathrm{e}^{\mathrm{i}\beta x}\,\mathrm{d}\alpha \tag{10}$$

$$u^{-}(x, y) = \int \mathcal{A}(\alpha)\frac{1}{2}\left(1 - \mu\frac{\beta_0}{\beta}\right)(1 - \kappa^2)\mathrm{e}^{\mathrm{i}\alpha y}\mathrm{e}^{-\mathrm{i}\beta x}\,\mathrm{d}\alpha. \tag{11}$$

Now we let h tend to infinity in order to recover the behaviour of the plane interface. From the Riemann–Lebesgue lemma, or else from weak convergence of $(\mathrm{e}^{\mathrm{i}(2\beta h)})$ towards zero, we get

$$\lim_{h\to+\infty} u^{+}(x, y; h) = 0 \tag{12}$$

and the same result holds for higher order rays (both up and down). At the diopter limit $h = +\infty$, only $u^{-}(x, y)$ exists in the Veselago medium. Now, we see that the transmitted beam inside the Veselago medium is a sum of plane waves $\exp[\mathrm{i}(\alpha y - \beta x)]$, and consequently we have to choose the minus sign convention, so that *we indeed have to change the usual radiation condition*: the wavevector $\boldsymbol{k} = (\alpha, -\beta)$ is in a sense opposite to that of the Poynting vector. In figure 1, we give a numerical example for a slab of electromagnetic parameters $\epsilon = -4$ and $\mu = -1$, embedded in vacuum, illuminated by a monochromatic Gaussian beam in s polarization. We have plotted the map of the electric field. We clearly see the negative refraction of the transmitted beam and the multiple beams inside the slab.

We have shown that provided that an isotropic homogeneous Veselago material can exist (at least for a certain frequency), then the Maxwell system leads naturally to the choice of a modified radiation condition inside this medium and to negative refraction. The previous analysis can be extended to the case of a quasimonochromatic beam of limited spatial extension.

References

[1] Veselago V G 1968 *Sov. Phys.–Usp.* **10** 509
[2] Pendry J B, Holden A J, Robbins D J and Stewart W J 1999 *IEEE Trans. Microw. Theory Tech.* **47** 2075
[3] Smith D R and Kroll N 2000 *Phys. Rev. Lett.* **85** 2933
[4] Shelby R A, Smith D R, Nemat-Nasser S C and Schultz S 2001 *Appl. Phys. Lett.* **78** 489
[5] Shelby R A, Smith D R and Schultz S 2001 *Science* **292** 77
[6] Landau L D and Lifschitz E M 1984 *Electrodynamics of Continuous Media* (Oxford: Pergamon)
[7] Pendry J B 2000 *Phys. Rev. Lett.* **85** 3966
[8] Williams J M 2001 *Phys. Rev. Lett.* **87** 249703
[9] 't Hooft G W 2001 *Phys. Rev. Lett.* **87** 249701
[10] Garcia N and Nieto-Vesperinas M 2002 *Phys. Rev. Lett.* **88** 207403
[11] Valanju P M, Walser R M and Valanju A P 2002 *Phys. Rev. Lett.* **88** 187401
[12] Ziolkowski R and Heyman E 2001 *Phys. Rev.* E **64** 056625
[13] Pokrovsky A I and Efros A L 2002 *Solid State Commun.* **124** 283
[14] Foteinopoulou S, Economou E N and Soukoulis C M 2002 *Preprint* cond-mat/0210346